

Sudden death and long-lived entanglement of two three-level trapped ions

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The dynamical properties of quantum entanglement in two three-level trapped ions interacting with a laser field are studied in terms of the negative eigenvalues of the partial transposition of the density operator. In contrast to the usual belief that destroying the entanglement can be observed due to the environment, it is found that the Stark shift can also produce sudden death of entanglement and long-lived entanglement between the qubits that are prepared initially in separable states or mixed states.

1 Introduction

As a promising resource, quantum entanglement plays a key role in quantum information processing such as quantum teleportation [1], superdense coding [2], and quantum key distribution [3]. However, in the real world, quantum information processing will be inevitably affected by the decoherence that destroys quantum superposition and quantum entanglement. The extent to which decoherence affects quantum entanglement is an interesting problem [4], and many researchers study it extensively based on various models [5, 6, 7, 8] with the point of view of environment induced decoherence.

The decay of entanglement cannot be restored by local operations and classical communications, that is one of the main obstacles to achieve a quantum computer [9]. Therefore it becomes an important subject to study the loss of entanglement [4, 11, 13, 14, 15, 16]. Quite recently, by using vacuum noise two-qubit, entanglement terminated abruptly in finite time has been performed [4] and the entanglement dynamics of a two two-level atoms model have been discussed [11, 12]. They called the non-smooth finite-time decay entanglement sudden death. Although entanglement can be realized in different ways in experiments, how we can preserve it is still a big challenge for current technology [17]. Because for open system, entanglement is fragile and decays exponentially, it is often thought as similar as quantum decoherence. Most of the authors who have treated this problem have dealt with the case in which the Stark shift has been ignored [4, 10]. However, in reality it cannot be ignored. The main aim of the present paper is try to answer the following question: what happens to two qubits entanglement when

we consider the Stark shift and different initial state setting in the absence or presence of the decoherence?

We present an explicit connection between the initial state setting, Stark shift and the dynamics of the entanglement. We give a condition for the existence of either entanglement sudden death or long-lived entanglement. In particular, a quantitative characterization of a general system of two three-level trapped ions interacting with a laser field is presented. We present various numerical examples in order to monitor the partial transpose of the density operator and entanglement dynamics. In principle, by proper adjustment of the initial state parameters, we can always find suitable values of Stark shift which can be use to suppress the decay of entanglement.

2 Model

The physical system on which we focus is a two three-level harmonically trapped ions with its center-of-mass motion quantized. The electronic levels $|a\rangle$, $|b\rangle$ and $|c\rangle$ are assumed to be metastable and coupled via a laser field of the form [18]

$$E(\hat{x}, t) = E_0 \exp[i(\hat{k} \cdot \hat{x} - \omega t + \phi(t))], \quad (1)$$

where E_0 is the strength of the electric field, \hat{k} is the wave vector of the driving laser field, \hat{x} is the position operator associated with the center-of-mass motion and ω is the laser frequency. We denote by $\phi(t)$ the fluctuations in the laser phase. Therefore, we can express the center-of-mass position in terms of the creation and annihilation operators of the one-dimensional trap namely $\hat{x} = \sqrt{\hbar/(2m\omega_s)}(\hat{a}^\dagger + \hat{a}) = \Delta x(\hat{a}^\dagger + \hat{a})$. We denote by \hat{a} and \hat{a}^\dagger the annihilation and creation operators and ω_s is the vibrational frequency related to the center-of-mass harmonic motion along the direction \hat{x} . In the absence of the rotating wave approximation, the trapped ions Hamiltonian may be written as

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \quad (2)$$

where $\hat{H}_0 = \hbar\omega_s\hat{a}^\dagger\hat{a} + \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b| + \hbar\omega_c|c\rangle\langle c|$, and

$$\hat{H}_1 = \hbar \sum_{i=1}^2 \left(\lambda_1^{(i)} e^{-i(k_i \cdot \hat{x} - \omega t + \phi_i)} S_{bc}^{(i)} + \lambda_2^{(i)} e^{-i(k_i \cdot \hat{x} - \omega t + \phi_i)} S_{ac}^{(i)} + c.c. \right). \quad (3)$$

The transition in the three-level ions is characterized by the dipole matrix element $\lambda_j^{(i)}$ where $S_{lm}^{(i)} = |l^{(i)}\rangle\langle m^{(i)}|$, $(l, m = a, b, c)$. For the sake of simplicity (but without loss of generality),

we have assumed to deal with the case in which $\phi_1 = 0$, $\phi_2 = \phi$ and the level $|c\rangle$ is assumed to be dipole-coupled to both the levels $|a\rangle$ and $|b\rangle$ via a far detuned laser field. While this is straightforward, it is often the case that it is simpler to work in the interaction picture in which the Hamiltonian (2) evolves in time according to the interaction with the vacuum field. If we express the center-of-mass position in terms of the creation and annihilation operators, the interaction part of equation (2) becomes

$$\begin{aligned}\hat{H}_{int} = & \hbar\Delta \sum_{i=1}^2 \left(S_{bb}^{(i)} + S_{aa}^{(i)} \right) \\ & + \hbar \left(\lambda_1^{(1)} e^{-i\eta(\hat{a}^\dagger + \hat{a})} S_{bc}^{(1)} + \lambda_2^{(1)} e^{-i\eta(\hat{a}^\dagger + \hat{a})} S_{ac}^{(1)} + c.c. \right) \\ & + \hbar \left(\lambda_1^{(2)} e^{-i\eta(\hat{a}^\dagger + \hat{a})} S_{bc}^{(2)} e^{-i\phi} + \lambda_2^{(2)} e^{-i\eta(\hat{a}^\dagger + \hat{a})} S_{ac}^{(2)} e^{-i\phi} + c.c. \right),\end{aligned}\quad (4)$$

where $\eta = k\sqrt{\frac{\hbar}{2M\omega_s}}$, is the Lamb-Dicke parameter. In equation (4) the time-dependent factor is eliminated in the interaction picture, since $\omega_c - (\omega_b + \Delta) = m\omega_s$ and $\omega_c - (\omega_a + \Delta) = m\omega_s$. We assume the Lamb-Dicke regime with small η . In order to obtain this we detune the laser frequency ω to the m^{th} vibrational red sideband. Also, we apply the rotating wave approximation discarding the rapidly oscillating terms and selecting the terms that oscillate with minimum frequency [19]. In these limits we can expand the interaction Hamiltonian to lowest order in η . The resulting Hamiltonian may be written as

$$\begin{aligned}\hat{H}_{int} = & \hbar\Delta \sum_{i=1}^2 (S_{bb}^{(i)} + S_{aa}^{(i)}) + \hbar(\zeta_1^{(1)} \hat{a}^{\dagger m} S_{bc}^{(1)} + \zeta_2^{(1)} \hat{a}^{\dagger m} S_{ac}^{(1)} + c.c.) \\ & + \hbar(\zeta_1^{(2)} \hat{a}^{\dagger m} S_{bc}^{(2)} e^{-i\phi} + \zeta_2^{(2)} \hat{a}^{\dagger m} S_{ac}^{(2)} e^{-i\phi} + c.c.),\end{aligned}\quad (5)$$

with a new coupling parameter $\zeta_i^{(j)}$ that includes the Dicke parameter in its definition. The analysis of such a Hamiltonian model can be carried out, providing eliminating of the non-resonantly coupled atomic level $|c\rangle$ adiabatically, due to the large detuning, the transitions for instance from the level $|a\rangle$ to the level $|c\rangle$ are very fast and immediately followed by decays on the atomic level $|b\rangle$. Therefore, considering only coarse grained observables, meaning that the system is observed at a rough enough time scale, effectively eliminates the far detuned level, namely, at such a time scale, the only observables and hence meaningful dynamical behaviors, involve levels $|a\rangle$ and $|b\rangle$ as a result of time averaging second order processes having $|c\rangle$ as an intermediate virtual level. This procedure then suppresses the fine dynamics, that is it sacrifices any information concerning the fast dynamics the third level is involved in. So that the effective Hamiltonian of the system including the ac-Stark shift, in the dipole and rotating

wave approximation, can be written as [20] ($\hbar = 1$)

$$\begin{aligned}\hat{H} = & \hat{a}^\dagger \hat{a} (\beta_1 S_{ba}^{(1)} S_{ab}^{(1)} + \beta_2 S_{ab}^{(1)} S_{ba}^{(1)}) + \hat{a}^\dagger \hat{a} (\beta_1 S_{ba}^{(2)} S_{ab}^{(2)} + \beta_2 S_{ab}^{(2)} S_{ba}^{(2)}) + \omega \hat{a}^\dagger \hat{a} \\ & + \omega (S_{aa}^{(1)} - S_{bb}^{(1)}) + \omega (S_{aa}^{(2)} - S_{bb}^{(2)}) + (\zeta_1 S_{ab}^{(1)} \hat{a}^2 + \zeta_2 e^{i\phi} S_{ab}^{(2)} \hat{a}^2 + c.c.).\end{aligned}\quad (6)$$

We denote by β_1 and β_2 the intensity-dependent Stark shifts, that are due to the virtual transitions to the intermediate relay level and $\zeta_i = \frac{\eta^2 \zeta_1^{(i)} \zeta_2^{(i)}}{\Delta}$.

A master equation that describes intrinsic decoherence [21, 22, 23] provides a general form for time evolution of the system density operator $\hat{\rho}(t)$. This master equation is written in the form

$$\frac{d}{dt} \hat{\rho}(t) = -i[\hat{H}, \hat{\rho}] - \frac{\gamma}{2} [\hat{H}, [\hat{H}, \hat{\rho}]], \quad (7)$$

where γ is the phase decoherence rate. The first term on the right-hand side of Eq. (7) generates a coherent unitary evolution of density matrix, while the second term represents the decoherence effect of the environment on the system and generates an incoherent dynamics of system. The equation with the similar form has been proposed to describe the intrinsic decoherence [24].

Under Markov approximations the solution of the master equation can be expressed in terms of Kraus operators [4] as follows

$$\begin{aligned}\hat{\rho}(t) &= \sum_{m=0}^{\infty} \frac{(\gamma t)^m}{m!} \hat{H}^m \exp(-i\hat{H}t) \exp\left(-\frac{\gamma t}{2} \hat{H}^2\right) \hat{\rho}(0) \exp\left(-\frac{\gamma t}{2} \hat{H}^2\right) \exp(i\hat{H}t) \hat{H}^m \\ &= \sum_{m=0}^{\infty} \frac{(\gamma t)^m}{m!} \hat{M}^m(t) \hat{\rho}(0) \hat{M}^{\dagger m}(t),\end{aligned}\quad (8)$$

where $\hat{\rho}(0)$ is the density operator of the initial state of the system and \hat{M}^m are the Kraus operators which completely describe the reduced dynamics of the qubits system,

$$\hat{M}^m = \hat{H}^m \exp(-i\hat{H}t) \exp\left(-\frac{\gamma t}{2} \hat{H}^2\right). \quad (9)$$

Equation (8) can also be written as

$$\hat{\rho}(t) = \exp(-i\hat{H}t) \exp\left(-\frac{\gamma t}{2} \hat{H}^2\right) \{e^{\hat{S}_M t} \hat{\rho}(0)\} \exp\left(-\frac{\gamma t}{2} \hat{H}^2\right) \exp(i\hat{H}t) \quad (10)$$

where we have defined the superoperator $\hat{S}_M \hat{\rho}(0) = \hat{H} \hat{\rho}(0) \hat{H}$. We will choose the following mixed state of the ions

$$\rho^a(0) = \cos^2 \theta |a, b\rangle \langle a, b| + \sin^2 \theta |b, a\rangle \langle b, a| \in \mathfrak{S}_A. \quad (11)$$

while the initial state of the vibrational mode is in a vacuum state $\rho^f(0) = |0\rangle \langle 0| \in \mathfrak{S}_F$. Then the initial state of the system can be written as $\hat{\rho}(0) = \rho^a(0) \otimes \rho^f(0)$.

3 Entanglement

In this paper we take the measure of negative eigenvalues for the partial transposition of the density operator. It was proved that the negativity is an entanglement monotone [25], hence, the negativity is a good entanglement measure. According to the Peres and Horodecki's condition for separability [26], a two-qubit state for the given set of parameter values is entangled if and only if its partial transpose is negative. The measure of entanglement can be defined in terms of the negative eigenvalues of the partial transposition in the following form [27]

$$I_\rho(t) = 2 \max \left(0, -\lambda_{neg}^{(i)} \right) \quad (12)$$

where $\lambda_{neg}^{(i)}$ is the sum of the negative eigenvalues of the partial transposition of the time-dependent reduced atomic density matrix ρ^a , which can be obtained by tracing out the field variables

$$\rho^a = Tr_f(\hat{\rho}(t)), \quad (13)$$

In the two qubit system ($C^2 \otimes C^2$) it can be shown that the partial transpose of the density matrix can have at most one negative eigenvalue [26]. The entanglement measure then ensures the scale between 0 and 1 and monotonously increases as entanglement grows. An important situation is that, when $I_\rho(t) = 0$ the two qubits are separable and $I_\rho(t) = 1$ indicates maximum entanglement between the two qubits.

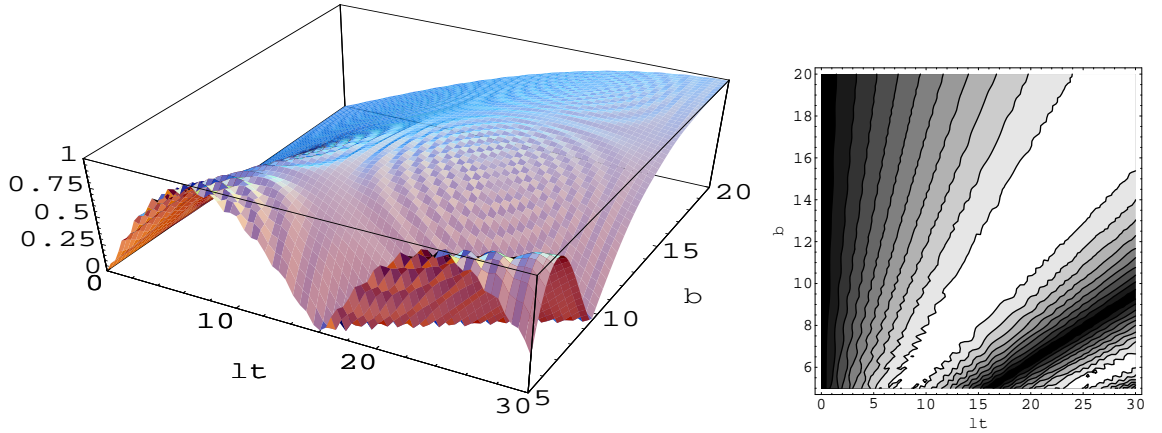


Figure 1: The evolution of the quantum entanglement $I_\rho(t)$ as a function of the scaled time and Stark shift parameter β . The parameters are $\bar{n} = 20$, $\Delta = 0$, $\theta = 0$ and $\gamma = 0$.

An interesting question is whether or not the entanglement is affected by the different parameters of the present system with the initial state in which one of the qubits is prepared

in its excited state and the other in the ground state. In particular, we focus on the effect of the mixed state parameter θ , the Stark shift parameter β , ($\equiv \beta_1/\beta_2$) and the decoherence. A numeric evaluation of the entanglement measure leads to the plot in figure 1. We consider the initial state of the two ions $\theta = 0$. In this case, we see that the entanglement is equal to zero in a periodic way for a small values of the Stark shift parameter, this period is increased with decreasing the parameter r . It is remarkable to see that with the value of the Stark shift parameter, $r = 20$ the entanglement is only zero for the initial period of the interaction time, while long lived entanglement is observed as the time goes on. In this case we can say that, when the system is allowed to evolve without applying a phase shift, the entanglement is a periodic function of time for small values of the Stark shift while long-survival entanglement can be obtained for larger values of Stark shift (see figure 1) . This is particularly because of the nonlinear nature of the coupling in this case (two-photon process).

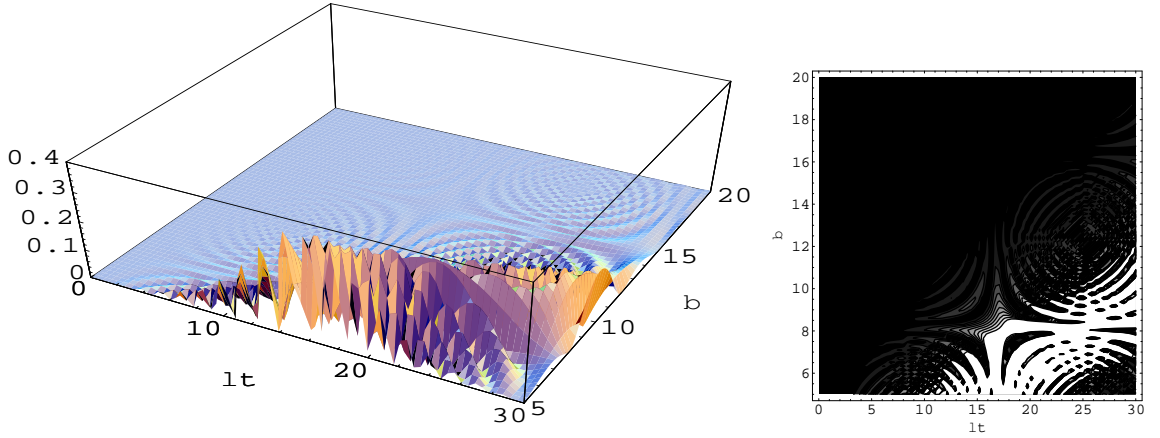


Figure 2: The evolution of the quantum entanglement $I_\rho(t)$ as a function of the scaled time and Stark shift parameter β . The parameters are $\bar{n} = 20$, $\Delta = 0$, $\theta = \frac{\pi}{4}$ and $\gamma = 0$.

In figure 2, we pause to touch on certain entanglement features when a mixed state of the qubits is considered as $\rho^a(0) = 0.5(|a, b\rangle\langle a, b| + |b, a\rangle\langle b, a|)$ i.e $\theta = \pi/4$. To this end we consider the same values of the other parameters similar to figure 1. The situation here is quite different from that observed in figure 1, where the entanglement exist only for small values of Stark shift while vanishes for all periods of the interaction time when $r > 12$. It is interesting to see here that entanglement sudden death is observed when the Stark shift is increased further (see figure 2). These properties show that the role played by the Stark shift on the entanglement is essential. Interestingly, when r is taken to be non zero, the values of the maximum entanglement are decreased, indicating that the mixed state setting leads to a

decreasing of the qubit-qubit entanglement. Generally speaking, because of the influence of mixed state parameter on entanglement, the amplitude of local maxima and minima decrease with increasing the deviation of r from the unity. However, as r takes values close to the unity we return to the same behavior in the initial pure state setting i.e $\rho = |a, b\rangle \otimes \langle a, b|$. However a slight change in r therefore, dramatically alters the entanglement. This is remarkable as the entanglement is strongly dependent on the initial state, which can be entangled or unentangled.

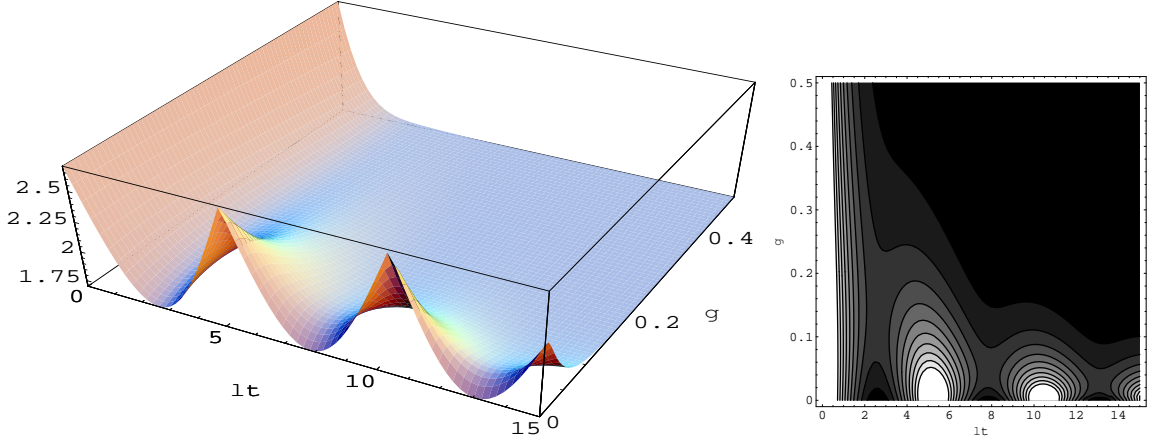


Figure 3: The evolution of the quantum entanglement $I_\rho(t)$ as a function of the scaled time and decoherence parameter γ . The parameters are $\bar{n} = 20$, $\Delta = 0$, $\theta = 0$ and $\beta = 0$.

We devote the discussion in figure 3 to consider the decoherence parameters effect on the entanglement in the absence of Stark shift. We would like to remark that decoherence due to normal decay is often said to be the most efficient effect in physics. Which means that, the entanglement increases rapidly, then approaches to a minimum value in a periodic manner. In this case, the entanglement introduced by the coherent interaction oscillates without dissipation, as showed in Fig. 3. Once, the environment has been switched on, i.e., $\gamma \neq 0$, it is very clear that the decoherence plays a usual role in destroying the entanglement. Also, from numerical results we note that with the increase of the parameter γ , a rapid decrease of the entanglement is shown (in agreement with [4]).

4 Conclusion

We have investigated the entanglement in the context of an ensembles of two identical qubits (or ions) and negativity as computable measure of the mixed-state entanglement has been used.

We have treated the more general case where the initial state of the two qubits can be mixed taking into account the presence of Stark shift. Through analysis, we find that the extent to which that the entanglement vanishes due to Stark shift relies not only on the Stark shift value, but also on the initial state setting. When the two ions start from a mixed state, the larger the Stark shift is, the faster the entanglement vanishes. For pure quantum states, the complete disentanglement occurs for a very short time in a periodic way only for small values of the Stark shift. We found that, the entanglement decay due to Stark shift for an initial mixed state is similar to the entanglement decay due to the decoherence. Finally, we expect our work will be helpful for preserving entanglement in practical experiments.

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References

- [1] G. Rigolin, Phys. Rev. A **71**, 032303 (2005); C.H. Bennett, et al., Phys. Rev. Lett. **70**, 1895 (1993);
- [2] C.H. Bennett, S.J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
- [3] M. Curty, M. Lewenstein, N. Lükenshaus, Phys. Rev. Lett. **92**, 217903 (2004); A.K. Ekert, Phys. Rev. Lett. **67**, 661 (1991);
- [4] T. Yu and J.H. Eberly, Phys. Rev. Lett. **97**, 140403 (2006); *ibid* Quantum Information and Computation, (2007) in press
- [5] S. I. Doronin, Phys. Rev. A **68**, 052306, (2003) .
- [6] V. Subrahmanyam, Phys. Rev. A **69**, 034304 (2004) .
- [7] X. S. Ma et al., Eur. Phys. J. D **137**, 135 (2006).
- [8] M. Asoudeh, V. Karimipour, Phys. Rev. A **70**, 052307 (2004).
- [9] C.H. Bennett, H.J. Bernstein, S. Popescu, B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
- [10] T. Yu, J. H. Eberly, Opt. Commun. **264**, 393 (2006)

- [11] M. Yonac, T. Yu, J. H. Eberly, J. Phys. B: At. Mol. Opt. Phys. **39**, S621 (2006)
- [12] J.-Q. Li, Z.-J. Li, C. Gang and J.-Q. Liang, Phys. Lett. A **359**, 275 (2006)
- [13] C. Pineda, T.H. Seligman, Phys. Rev. A **73**, 012305 (2006).
- [14] T. Yu, J.H. Eberly, Phys. Rev. B **66**, 193306 (2002)
- [15] T. Yu, J.H. Eberly, Phys. Rev. B **68**, 165322 (2003)
- [16] T. Yu, J.H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
- [17] C. P. Williams, and S. H. Clearwater, Explorations in Quantum Computing (1998) (New York: Telos, Springer-Verlag).
- [18] A. Messina, S. Maniscalco and A. Napoli, J. Mod. Opt. 50, 1 (2003); H.-I. Yoo, J. H. Eberly, Phys. Rep. **118**, 239 (1985); B. W. Shore and P. L. Knight, J. Mod. Opt. **40**, 1195 (1993)
- [19] I. Tittonen, S. Stenholm and I. Jex, Opt. Commun. **124**, 271 (1996).
- [20] I. K. Kudryavtsev, A. Lambrecht, H. Moya-Cessa and P. L. Knight, J. Mod. Opt. **40**, 1605, (1993); I. Jex, Quantum Optics **2**, 433 (1990); J. Mod. Opt. **39**, 835 (1990); I. K. Kudryavtsev and P. L. Knight, J. Mod. Opt. **40**, 1673, (1993); M. Abdel-Aty, J. Opt. B: Quantum Semiclass. Opt. **5**, 349 (2003); M. M. Ashraf, J. Opt. B: Quantum Semiclass. Opt. **3**, 39 (2001).
- [21] H.-P. Breuer and F. Petruccione The theory of open quantum systems, Oxford University Press, Oxford, (2002)
- [22] D.A. Lidar and K. B. Whaley, in Irreversible quantum dynamics edited F. Benatti and R. Floreanini, Springer Lecture Notes in Physics, Vol. 62, Berlin (2003), p.83.
- [23] C.W. Gardiner and P. Zoller, Quantum Noise (Springer-Verlag, Berlin, 2000).
- [24] G.J. Milburn, Phys. Rev. A **44**, 5401 (1991); S. Schneider and G.J. Milburn, Phys. Rev. A **57**, 3748 (1998); S. Schneider and G.J. Milburn, Phys. Rev. A **59**, 3766 (1999).
- [25] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002); W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

- [26] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996)
- [27] J. Lee and M. S. Kim, Phys. Rev. Lett. **84**, 4236 (2000); J. Lee, M. S. Kim, Y. J. Park and S. Lee, J. Mod. Opt. **47**, 2151 (2000).